Solitary waves and their critical behavior in a nonlinear nonlocal medium with power-law response

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We discuss a nonlocal generalization of the nonlinear Schrödinger equation and study propagation of solitary waves in a nonlinear nonlocal medium at its critical state, the response of which obeys the power law with the exponent k. Using the time-dependent variational principle, we derive a set of dynamical equations and develop the fixed-point analysis. A critical behavior is found in the k dependence of the width of the wave packet. We also present a proof of the stability of the system and discuss an oscillatory phenomenon in the self-focusing process. [S1063-651X(98)11104-2]

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Studies of nonlinear waves and theory of critical phenomena have long histories individually. However, wave dynamics in media with large response lengths seems to be still infant as a research area. There exist interesting problems to be addressed from the viewpoint of solitons. For example, when the width of a wave packet is comparable with or shorter than the characteristic response length of a medium, effects of nonlocality are expected to become significant [1]. Then the following questions naturally arise: Are there still solitonlike solutions? If yes, how is the medium nonlocality reflected in the physical properties of such localized waves?

The purpose of this paper is to discuss propagation of waves in a nonlinear medium at its critical state using a simple one-dimensional model. In this model, the propagation is governed by the nonlocal extension of the nonlinear Schrödinger equation (NLSE) [2]. Specifically the medium is characterized by its response obeying the power law with the exponent k. Since such a medium has no finite scale of response length, the influence of nonlocality is always relevant to any localized waves. We analyze this model based on the standard variational principle and identify a solitary-wave solution. We find a critical behavior in the k dependence of the width of the wave. We also present a proof of the stability of the system.

The model we consider is microscopically described by the action integral of the following form:

$$I[\psi^*,\psi,\sigma]=\int\int dt\; dx\left\{\frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)-\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}+\sigma\psi^*\psi+\mathcal{L}_\sigma\left(\sigma,\frac{\partial\sigma}{\partial t},\frac{\partial\sigma}{\partial x},\ldots\right)\right\},\tag{1}$$

where $\psi(x,t)$ and $\sigma(x,t)$ denote a complex wave field and a real local field describing the medium material, respectively. The spatial integrations are performed over the whole range $(-\infty,\infty)$. All quantities and parameters are made dimensionless for simplicity. $\mathcal{L}_\sigma$ stands for the Lagrangian density of the σ field. From this action, the field equation for ψ reads

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\sigma\psi=0.\tag{2}$$

The field equation for σ has the following form: $F(\sigma,\sigma_t,\sigma_{tt},\sigma_x,\sigma_{xx},\ldots)+\psi^*\psi=0$ ($\sigma_t\equiv\partial\sigma/\partial t$ and so on). The Lagrangian density $\mathcal{L}_\sigma$ and the corresponding field-equation function F have complicated structures microscopically, in general. However, in the picture of wave propagation in a background medium, what is needed is not the microscopic field σ itself but rather its statistical average $\langle\sigma\rangle$: a wave, whose typical wavelength is larger than the characteristic scale of a material structure of the medium, feels only $\langle\sigma\rangle$. Therefore a general response-theoretic discussion may apply. In the linear response theory, the average of σ with an external source is generically not equal to that without sources. The difference between the two can be written as follows [3]:

$$\overline{\sigma}(x,t)=\langle\sigma(x,t)\rangle_{\rm ext}-\langle\sigma(x,t)\rangle_0=\int dx'\;R(x-x')|\psi(x',t)|^2,\tag{3}$$

where $\langle\sigma\rangle_{\rm ext}$ and $\langle\sigma\rangle_0$ mean the statistical averages of σ with and without the source $|\psi|^2$, respectively. $\langle\sigma\rangle_0$ is a constant, in general. $R(x-x')$ is a response function of the medium and is essentially equivalent to the two-point correlation function of σ. A possible time dependence in R is not taken into account. That is, the relaxation time is assumed to be negligibly short.

Thus, the effective wave equation for ψ is obtained by replacing σ in Eq. (2) by $\langle\sigma\rangle_{\rm ext}$ as follows:

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+\int dx' R(x-x')|\psi(x',t)|^2\psi(x,t)=0,\tag{4}$$

provided that the constant part $\langle\sigma\rangle_0$ has been eliminated by redefining the phase of ψ. Obviously, Eq. (4) becomes the standard NLSE in the particular case of the singular response $R(x-x')\sim\delta(x-x')$.

Before proceeding, we wish to make brief comments on the integrodifferential equation (4). This equation is familiar in the Hartree-Fock theory for quantum many-body problems [4]. In the area of classical physics, on the other hand, this is of a very modern issue in nonlinear wave dynamics. For example, Engelbrecht [5] discusses that the nonlocal equations of this type of inhomogeneity appear in the problems of wave propagation in heterogeneous media, which are the media composed of superposition of several homogeneous media like a void distribution or a porous structure.

Now, suppose the medium is at its critical state. In this case, the response length becomes very large, ideally infinite [6,7]. To model such a medium, it is natural to assume the following power-law response [7]:

$$R(x-x')=\gamma|x-x'|^{-k} \quad (0<k<1), \tag{5}$$

where γ is a positive constant, and plays the role of a coupling constant describing the strength of nonlinearity of Eq. (4). This is a homogeneous function of degree $-k$: $R(\alpha x,\alpha x')=\alpha^{-k}R(x,x')$. k is simply called the exponent here. As mentioned above, it has no finite length scales, and therefore localized waves always feel the nonlocality in any situation. In what follows, first we discuss the stability of the system and then study the problem of a solitary wave propagating in the nonlinear nonlocal medium characterized by the response (5).

First we prove the stability of the system described by Eqs. (4) and (5). For this purpose, let us consider the following effective action:

$$I_{\mathrm{eff}}[\psi^*,\psi]=\int\int dt\ dx\left\{\frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right)-\left|\frac{\partial\psi}{\partial x}\right|^2\right\}+\frac{\gamma}{2}\int\int\int dt\ dx\ dx'|\psi(x,t)|^2|x-x'|^{-k}|\psi(x',t)|^2. \tag{6}$$

From this, follows the Hamiltonian

$$H=\int_{-\infty}^{\infty}dx\left|\frac{\partial\psi}{\partial x}\right|^2-\frac{\gamma}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\ \frac{|\psi(x,t)|^2|\psi(x',t)|^2}{|x-x'|^k} \quad (0<k<1). \tag{7}$$

The wave equation (4) with (5) can be derived from the canonical equation $i\partial\psi/\partial t=\delta H/\delta\psi^*$. To establish the general stability of the system, we have to show that the Hamiltonian is bounded below. First we note the following inequality (Theorem 382 in Ref. [8]):

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\ \frac{f(x)g(x')}{|x-x'|^k}\leqslant K(p,q)F^{1/p}G^{1/q} \quad \left(p,q>1,\frac{1}{p}+\frac{1}{q}>1,\ 0<k=2-\frac{1}{p}-\frac{1}{q}<1\right), \tag{8}$$

where K is a positive constant depending only on the parameters p and q, and

$$F=\int_{-\infty}^{\infty}dx\, f^p(x), \quad G=\int_{-\infty}^{\infty}dx\, g^p(x). \tag{9}$$

This inequality holds for any nonnegative function f and g, which belong to $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, respectively, where $L^p(\mathbb{R})=\{f|\int_{-\infty}^{\infty}dx\, f^p(x)<\infty\}$. We assume that the local intensity of a physical wave function of the system is a member of $L^p(\mathbb{R})$. Then, identifying f and g with $|\psi(x,t)|^2$ and setting $p=q$, we have

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\ \frac{|\psi(x,t)|^2|\psi(x',t)|^2}{|x-x'|^k}\leqslant K(p,p)\left(\int_{-\infty}^{\infty}dx|\psi(x,t)|^{2p}\right)^{2/p}. \tag{10}$$

Using this inequality together with the nonnegativity of the first term on the right-hand side of Eq. (7), we find

$$H\geqslant-\frac{\gamma}{2}K(p,p)\left(\int_{-\infty}^{\infty}dx|\psi(x,t)|^{2p}\right)^{2/p} \quad \left(p=\frac{2}{2-k}\right), \tag{11}$$

which proves the stability of the system.

Now we proceed to study the problem of solitary wave propagation. It is unlikely that exact solutions can be found for this system with a general k. The methods of inverse scattering or Lax pairs do not seem to work. Therefore we here examine the time-dependent variational principle in the Rayleigh-Ritz manner [9].

We optimize the functional (6) by employing the Gaussian ansatz for the trial wave function:

$$\psi(x,t)=\left[\frac{M^2}{2\pi G(t)}\right]^{1/4}\exp\left\{-\left[\frac{1}{4G(t)}-i\Lambda(t)\right][x-q(t)]^2+ip(t)[x-q(t)]\right\}, \tag{12}$$

where the total intensity of ψ is set to be

$$M=\int_{-\infty}^{\infty}dx|\psi(x,t)|^2. \tag{13}$$

M is constant in t since the action integral (6) [and, therefore, also Eq. (4)] is invariant under the global gauge transformation: $\psi\rightarrow e^{i\theta}\psi$, $\psi^*\rightarrow e^{-i\theta}\psi^*$. Substituting Eq. (12) into Eq. (6) and neglecting irrelevant total derivatives with respect to t, we have the reduced action

$$I_{\text{eff}}=\int dt\ L, \tag{14}$$

where L is the Lagrangian in the first-order form

$$L=M(p\dot{q}+\Lambda\dot{G})-H, \tag{15}$$

with the Hamiltonian

$$H=M\left[p^2+4G\Lambda^2+\frac{1}{4G}-\frac{\rho}{2}(4G)^{-k/2}\Gamma\left(\frac{1-k}{2}\right)\right], \tag{16}$$

provided that we have set

$$\rho=\frac{\gamma M}{\sqrt{\pi}}. \tag{17}$$

In the above, $\Gamma(z)$ is the gamma function of argument z and the overdot stands for the differentiation with respect to t. The reduced system has a simplectic structure in terms of the canonical variables (q,G,p,Λ), and thus the problem is translated into particle dynamics of these collective variables.

Now, Hamilton's principle of least action leads to the following set of equations:

$$\dot{q}=2p, \tag{18}$$

$$\dot{p}=0, \tag{19}$$

$$\dot{G}=8\Lambda G, \tag{20}$$

$$\dot{\Lambda}+4\Lambda^2=\frac{1}{4G^2}-\frac{C(k)}{4G^{1+k/2}}, \tag{21}$$

where $C(k)$ is a positive quantity given by

$$C(k)=\rho\,\frac{k}{2^k}\,\Gamma\left(\frac{1-k}{2}\right). \tag{22}$$

From Eqs. (18) and (19), it follows that the location of the peak of the wave packet has a motion of constant velocity: $q=2p_0t+q_0$, $p=p_0$ (q_0 and p_0 are arbitrary constants). Clearly this uniformity comes from the fact that the response (5) possesses the translational invariance. Henceforth, p_0 is set equal to zero:

$$q=q_0, \tag{23}$$

$$p=0. \tag{24}$$

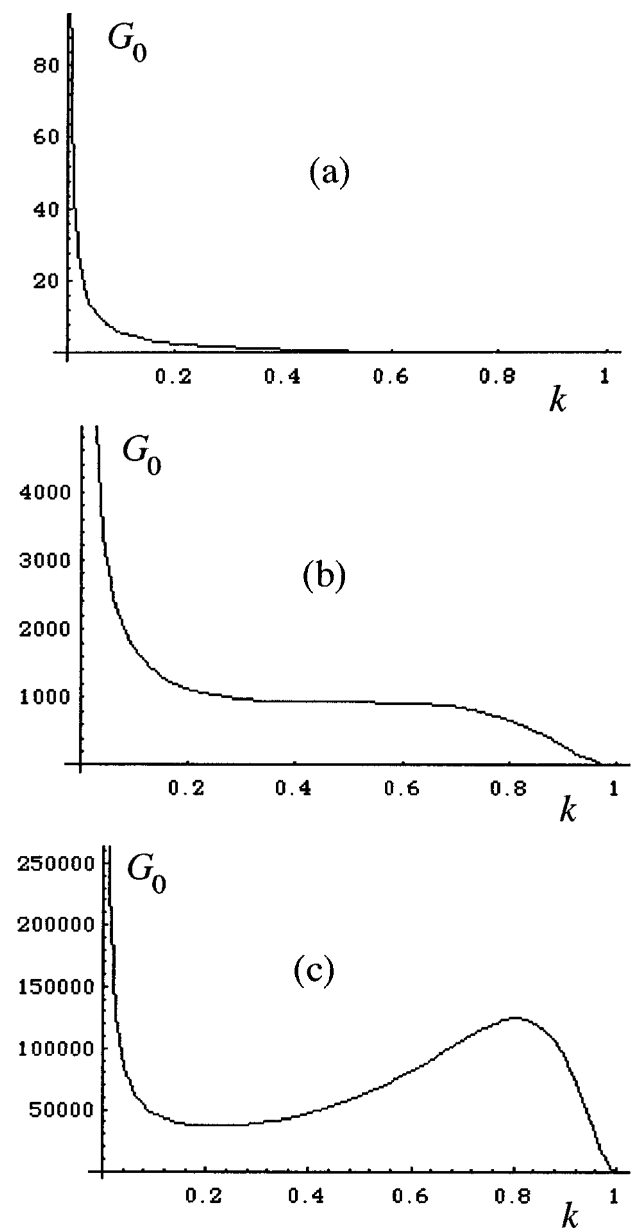

FIG. 1. Plots of G_0 as the function of the exponent k. The values of the parameter ρ are, respectively, (a) $\rho=1$, (b) $\rho=0.00462$, (c) $\rho=0.0002$. All quantities are dimensionless.

In contrast to the pair of (q,p), the motions of (G,Λ) are determined by the coupled nonlinear equations. We first perform the fixed-point analysis of the system of equations (20) and (21). The fixed point of these equations, which is the singular solution, is found to be

$$G_0(k)=C(k)^{2/(k-2)}, \tag{25}$$

$$\Lambda_0=0. \tag{26}$$

The wave function (12) with these parameter values describes a variational solitary wave, whose shape (or width) is kept unchanged through the propagation process. We note that Eq. (26) removes the chirp factor from the wave function at the fixed point.

In Fig. 1, we present the plots of G_0 with respect to the exponent k. Very different behaviors appear for large and small values of the parameter ρ in Eq. (17). The critical value of ρ is found to be

$$\rho_c = 0.00462 \ldots . \tag{27}$$

Above ρ_c the (squared) width of the solitary wave decreases monotonically as the exponent k grows, whereas below ρ_c nontrivial extrema are observed. It can be analytically shown that there exists only one local minimum. For $\rho > \rho_c$, the width decreases monotonically with respect to k. A larger value of k corresponds to a stronger singularity in the response, which may give a tight mode coupling to realize a sharp focusing. However, for $\rho < \rho_c$, the situation drastically changes. For example, in a medium having $k \cong 0.8$, the width at the fixed point is almost three times larger than that in a medium of $k \cong 0.2$. Therefore, below ρ_c, the sharp focusing is hardly obtained in media with relatively large k. This is the main result of the present work.

Finally we briefly discuss small perturbation on the variational wave packet. Let us consider the initial condition. Since our system is an autonomous Hamiltonian system, the Hamiltonian (16) is a constant of motion. It is set as

$$H = ME, \tag{28}$$

where E is a constant to be determined by an initial condition. Therefore, recalling Eq. (24), it follows from this constraint that

$$\dot{G}^2 = 16EG - 4 + \frac{8C(k)}{k} G^{1-k/2}. \tag{29}$$

Changing the variable

$$G = 2Q^2, \tag{30}$$

we can rewrite Eq. (29) in the standard mechanical form:

$$\frac{1}{2}\dot{Q}^2 = E - U(Q), \tag{31}$$

where U is the potential given by

$$U(Q) = \frac{1}{8Q^2} - \frac{C(k)}{2^{1+k/2} k Q^k}. \tag{32}$$

This potential is bounded below and approaches zero from the negative values. Its minimum value is determined by the fixed-point solution:

$$E_{\min} = U(\sqrt{G_0/2}). \tag{33}$$

Thus, with any initial condition satisfying

$$E_{\min} \leqslant E < 0, \tag{34}$$

the motion of Q is bounded and oscillatory, and therefore the solution is in fact stable. In particular, the oscillation with frequency

$$\omega = \sqrt{2-k}\, C(k)^{2/(2-k)} \tag{35}$$

is induced by small perturbation on the fixed point, i.e., the variational shape-preserving solution.

In conclusion, we have studied the physical properties of a solitary wave propagating in the nonlinear nonlocal medium with the power-law response. Using the time-dependent variational principle with the Gaussian ansatz, we have clarified how the width of the solitary wave depends on the exponent of the power law. We have found a critical behavior of the wave-packet width under the change of the value of the coupling constant multiplied by the total intensity. We have also presented the proof of the stability of the system for a general configuration of physical wave functions.

We have studied a nonlocal extension of the NLSE. The standard NLSE is known to play a central role in nonlinear optics and optical solitons [2]. [There, the time parameter should be replaced by the coordinate of the longitudinal (propagation) axis.] The nonlinearity is induced by the Kerr media. The present work suggests that it is of interest to investigate optical properties of such media at their critical states from the viewpoint of the quality of self-focusing of optical solitons.

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